

# Finite-Element Analysis of High-Frequency Axisymmetric Device Problems

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**Abstract**—A finite-element scheme for solving wave propagation problems in devices with complete azimuthal symmetry both in their material properties and geometry is presented. The  $\phi$ -coordinate variation is restricted to  $\sin(n\phi + \xi)$ , thus requiring only a two dimensional space for the complete finite-element analysis. The open radiating boundaries inherent in many propagation problems are tackled through simulated absorption of the radiation by lossy materials. Also proposed for these kinds of boundaries is a hybrid FEM-BEM formulation, based on the vector Green's function for the Helmholtz equation. This method has been tested by application to an open-ended circular waveguide and a comparison with the simulated-absorption results.

## I. INTRODUCTION

The finite-element solution to high-frequency electromagnetic problems is characterised by high computing demands as indicated by Webb et al [1]. Recent research has therefore tended to focus on 2-D problems, which in some cases are worked out through one field component [2]. Since all problems are 3-D in nature, some aspect of the problem, e.g. the symmetry, is usually exploited so as to allow solution in a 2-D space. In many cases, the field variables are assumed to be constant in one of the coordinate directions. This paper deals with the exploiting of symmetry to solve axisymmetric problems, arising from configurations with complete azimuthal symmetry in all material properties and geometry, through a 2-D space.

The variational formulation for non-self-adjoint electromagnetic problems with various boundary conditions was derived by Chen and Lien [3]. They showed that when lossy anisotropic materials are present, the functional

$$F(\mathbf{H}) = \int_{\Omega} \left[ (\nabla \times \mathbf{H})^T \cdot (\epsilon_r^{-1} \nabla \times \mathbf{H}) - k_0^2 \mathbf{H} \cdot \mu_r \mathbf{H} \right] d\Omega - \int_S \mathbf{H} \times (\epsilon_r^{-1} \nabla \times \mathbf{H}) \cdot d\mathbf{S} \quad (1)$$

is stationary about the correct magnetic field solution,  $\mathbf{H}$ . A close observation of this functional shows that when only a  $\sin(n\phi + \xi)$  variation is allowed in the  $\phi$ -coordinate direction in a  $(\rho, \phi, z)$  cylindrical coordinate system with  $n$  being an integer, the  $\phi$ -dependence in (1) integrates out. The finite-element analysis is therefore required to be carried out only in the  $\rho - z$  plane. This  $\sin(n\phi + \xi)$  variation is characteristic in many circular waveguiding structures such as the cylindrical waveguide and the circular corrugated horn [4].

For wave problems with open radiating boundaries, several approaches have lately developed [5]. Hirayama and Koshiba [6], for example, have used the hybrid FEM-BEM method in their analysis of open dielectric slab waveguide. More recently, Webb [7] has employed the second-order absorbing condition [8] to solve 2-D wave problems by enclosing the device with a circular boundary. In this paper, a vector analysis of coupling the FEM and BEM is carried out and a method that uses a lossy medium to simulate absorption of the radiated fields is also presented.

## II. THE FINITE-ELEMENT TREATMENT OF THE PROBLEM

First, the expression for the magnetic field,

$$\mathbf{H} = \bar{\rho} H_{\rho}(\rho, z) \sin(n\phi) + \bar{\phi} H_{\phi}(\rho, z) \cos(n\phi) + \bar{z} H_z(\rho, z) \sin(n\phi), \quad (2)$$

where the integer  $n$  is known, is substituted into the functional in (1). For the  $\phi$ -variation in (2) to hold,  $\epsilon_r$  and  $\mu_r$  should take the form

$$\epsilon_r = \begin{pmatrix} \epsilon_1 & -j\epsilon_2 & 0 \\ j\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \quad \mu_r = \begin{pmatrix} \mu_1 & -j\mu_2 & 0 \\ j\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}. \quad (3)$$

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The region defined by  $\Omega_1$ ,  $\Omega_2$  and  $\Omega_3$  in Fig.1 is then discretised into 8-node isoparametric quadrilateral elements [9].

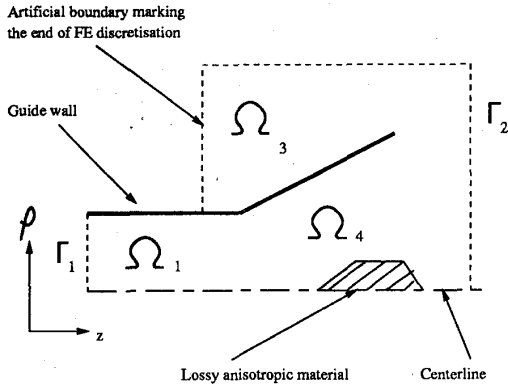


Fig. 1. A general radiating waveguide problem

The trial  $\mathbf{H}$ -field solution within each element is then set up through the eight unknown nodal field vectors and corresponding scalar functions. The transverse components  $\mathbf{H}_\rho$  and  $\mathbf{H}_\phi$  of the dominant  $\text{TE}_{11}$  mode are then prescribed at the input plane  $\Gamma_1$ , whilst applying the Homogeneous Neuman boundary [9] condition at the conducting walls.

### III. SIMULATED ABSORPTION BY A LOSSY MEDIUM

Materials with complex permeabilities  $\mu_r$  and permittivities  $\epsilon_r$  have been used to absorb electromagnetic radiation both in practice [10] and in field modelling problems [11]. This absorption is simulated here for the open boundaries, with  $\mu_r$  and  $\epsilon_r$  being chosen to minimise reflections at the material interface, while maximising absorption inside the material.

The presence of a simulated lossy border around the problem space will in general create unwanted reflections. These can be minimised to some extent by choosing the lossy  $\frac{\mu_r}{\epsilon_r}$  equal to unity, when radiation impinging normally upon the artificial boundary shown in Fig.1 is perfectly absorbed. Since normal incidence is difficult to guarantee, as well, we have kept the complex  $\mu_r$  and  $\epsilon_r$  close to unity and employed a sufficient thickness of the lossy cladding.

### IV. VECTOR BOUNDARY-ELEMENT FORMULATION

An artificial boundary  $\Gamma_2$  in the open unbounded region, as in Fig.1, is placed close to the open end of the guide. For unity  $\mu_r$  and  $\epsilon_r$ , Maxwell's equations result in the homogeneous Helmholtz equation

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}) - k_0^2 \mathbf{H}(\mathbf{r}) = 0, \quad (4)$$

where  $k_0 = \sqrt{\omega^2 \mu_0 \epsilon_0}$  is the wavenumber. Further, the vector Green's function satisfies the inhomogeneous Helmholtz equation

$$\nabla \times \nabla \times \mathbf{G}(\mathbf{r}, \mathbf{r}') - k_0^2 \mathbf{G}(\mathbf{r}, \mathbf{r}') = -\delta(|\mathbf{r} - \mathbf{r}'|) \bar{\mathbf{a}}, \quad (5)$$

where  $\bar{\mathbf{a}}$  is an arbitrary unit vector. It can then be shown that [12]

$$\int_{S+S_\infty} [\mathbf{H}(\mathbf{s}) \times \nabla \times \mathbf{G}(\mathbf{s}, \mathbf{r}') - \mathbf{G}(\mathbf{s}, \mathbf{r}') \times \nabla \times \mathbf{H}(\mathbf{s})] \cdot \mathbf{n} dS = \frac{1}{2} \bar{\mathbf{a}} \cdot \mathbf{H}(\mathbf{r}'), \quad (6)$$

with  $\mathbf{n}$  being an outward normal vector to the boundary.

The field  $\mathbf{H}$  is confined to the expression in (2) and a second-order interpolation carried out along  $\Gamma_2$ . The discretisation yields the partitioned matrix equation

$$\begin{pmatrix} [M]_1 & [M]_2 & [M]_3 \end{pmatrix} \begin{pmatrix} [H]_\rho \\ [H]_\phi \\ [H]_z \end{pmatrix} = \begin{pmatrix} [M]_4 & [M]_5 & [M]_6 \end{pmatrix} \begin{pmatrix} [q]_\rho \\ [q]_\phi \\ [q]_z \end{pmatrix}, \quad (7)$$

where the vectors  $[q]_\rho$ ,  $[q]_\phi$  and  $[q]_z$  contain the the outward normal differentials  $\frac{\partial H_\rho}{\partial \mathbf{n}}$ ,  $\frac{\partial H_\phi}{\partial \mathbf{n}}$  and  $\frac{\partial H_z}{\partial \mathbf{n}}$  respectively. This matrix is then coupled to the FEM matrix as in [12] and [6].

### V. APPLICATION TO WAVEGUIDE PROBLEMS

#### A. The Closed Waveguide Problem

Our finite-element analysis was tested and validated by solving the problem of a closed waveguide terminated with a plug of lossy material. The test was carried out as in [13] with the problem possessing the axisymmetry corresponding to (2). Less than 0.1% error, compared with the analytical result, was found in impedance calculations for frequencies ranging between the cut-off and resonance for TE and TM modes from 01 to 33.

#### B. The Radiating Cylindrical Waveguide

A cylindrical waveguide with infinitely thin walls was terminated abruptly, thus radiating into the free space, see Fig.2. A  $\text{TE}_{01}$  mode was made incident at the input plane and the impedance calculated by the finite-element method, with lossy absorbing materials simulating an anechoic chamber. This impedance is referred to the plane of the truncation T, so that the results can be compared with those in the waveguide handbook [14].

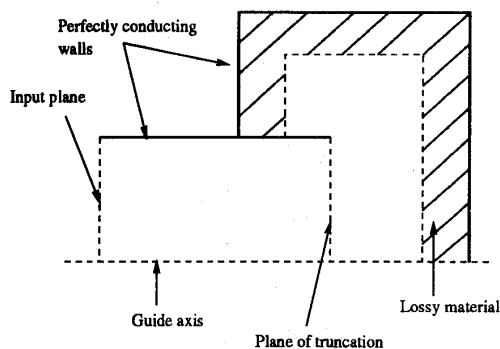


Fig.2. Abruptly terminated waveguide with absorbing material

The impedance plot is made for a range of the waveguide radii,  $\frac{r}{\lambda_0}$ , in Fig.3(a) and (b), where the comparison is made with the handbook values.

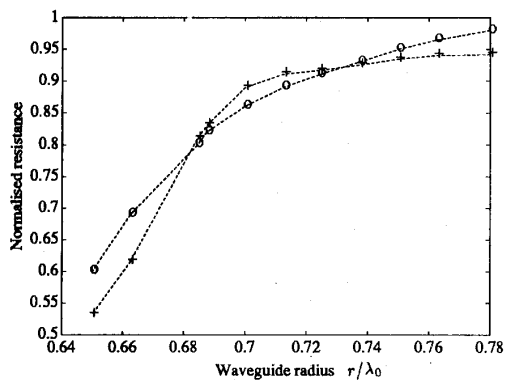


Fig.3(a). Plot of the normalised resistance at the plane of truncation T, (+ represents simulated-absorption values, o represents handbook values)

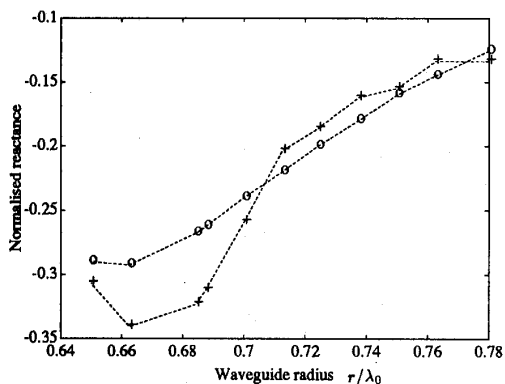


Fig.3(b). Plot of the normalised reactance at the plane of truncation T, (+ represents simulated-absorption values, o represents handbook values)

C. Application of the hybrid FEM-BEM to the Truncated Circular Waveguide Problem

To check the hybrid FEM-BEM proposal, the truncated waveguide problem was now solved using this method and the results compared with those of the absorption technique. The normalised  $TE_{11}$  mode input impedance was calculated for several values of the wavenumber,  $k_0$ . The values of the normalised impedance, referred to the plane of truncation, T, are compared in Fig.4(a) and (b).

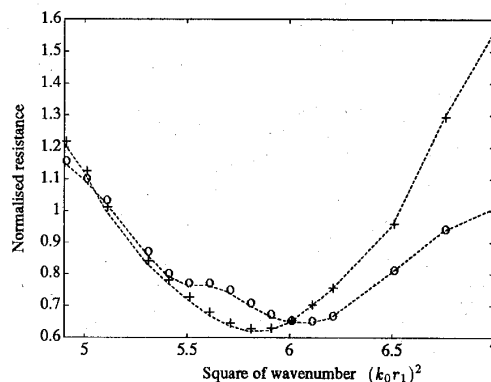


Fig.4(a). Plot of the normalised resistance at the plane of truncation T, (o represents simulated-absorption values, + represents FEM-BEM values)

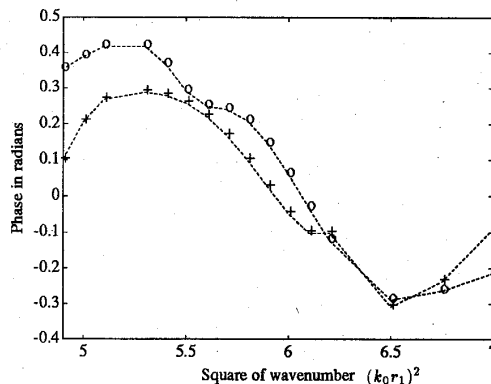


Fig.4(b). Plot of the normalised reactance at the plane of truncation T, (o represents simulated-absorption values, + represents FEM-BEM values)

D. The Circular Cross-Section Corrugated Waveguide

The configuration shown in Fig.5 is commonly used for exciting a  $HE_{11}$  mode in a circular cross-section corrugated waveguide, with the transition from the smooth-walled cylindrical waveguide being designed to achieve maximum mode conversion. This problem was analysed

by Clarricoats and Saha [15], through the space-harmonic formulation, and by James [16] using the mode matching technique.

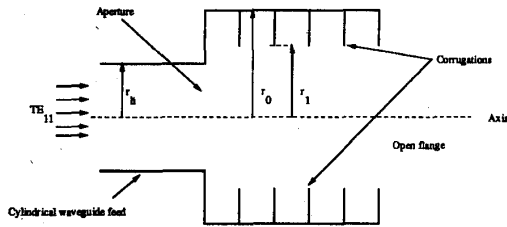


Fig.5. A circular cross-section corrugated waveguide with feed

In this paper, the input impedance of the structure in Fig.5 is determined using the finite-element method, with the radiation from the open flange being handled through the simulated absorption technique. The values of the reflection coefficient  $\rho$  at the junction are compared in Fig.6 with those by Clarricoats [15] for a range of frequencies.

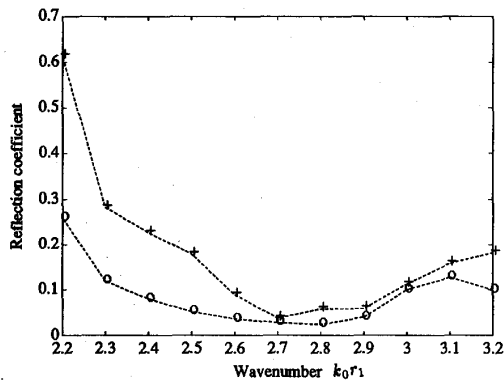


Fig.6. Plot of reflection coefficient  $\rho$  against the wavenumber  $k_0$  for  $r_1/r_h = 1.1$  and  $r_1/r_0 = 0.5$  (+ represents FE values, o represents values in [5])

## VI. CONCLUSION

A finite-element scheme for solving axisymmetric problems through a 2-D space has been presented and verified. The method can be applied to problems with axial discontinuities as illustrated through the circular corrugated waveguide problem. The nature of the permittivity and permeability-tensors used, implies that the method can also be applied to devices containing ferrites and plasmas. For the two open boundary-methods presented, the absorption technique is preferred as it entails solving a

purely finite-element problem, whereas the use of boundary elements involves the storage of a non-symmetrical dense matrix which greatly increases the cost of the solution.

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